Role of membranes in hydrodynamic interaction of small particles

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The response of a fluid to a stationary force in the presence of membranes has been studied. Two different geometries of membrane systems are considered: two parallel flat membranes and a nearly spherical vesicle. In the case of a force acting between two parallel membranes, an induced velocity both between and behind the membranes is found. Between the membranes, at large distances from the application point, the induced velocity decays with the distance as $1/r$, more slowly than between two solid walls where it decays as $1/r^2$, and similar to the decay in an unconfined liquid. Behind the membrane interface, the flow does not have a component normal to the membranes, and the normal component of the force does not affect the flow in this region. In the case of spherical symmetry, expressions for the flow both inside and outside the membrane vesicle are found in terms of spherical harmonics. We discuss the applications of our results to Brownian motion of particles in the system and the possibility of measuring the membrane internal viscosity.

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I. INTRODUCTION

The hydrodynamic interaction of particles immersed in a fluid plays an essential role in the physics of suspensions, where it determines the dynamic properties on scales of the order of the interparticle distance. In spatially confined suspensions, boundary effects can play an essential role in the particles' interactions. There have been a number of reports on the physics of suspensions and the particle interactions in different conditions: particles near a wall [1], between two parallel walls [2,3], and near an interface of immiscible liquids [4]. In the present work, we undertake theoretical research into the particle interactions in dilute suspensions in the presence of membranes.

We consider the response of a fluid in a membrane system to a force by solving the hydrodynamic equations. The force is supposed to be sufficiently small that the flow caused by it can be described by the Stokes equation. For the same reason, we neglect the membrane deformations caused by the flow when applying boundary conditions, as with gravity and the capillary-wave problem (see [5], Secs. 12 and 25). We solve the Stokes equation with a stationary point force acting on the fluid. The response to an arbitrarily distributed force can then be found by means of integration. The solution is also applicable for slowly varying forces, provided their frequencies are much less than the reciprocal characteristic viscous relaxation time of the system. The solution of the equation with the point force can also be used to find the fluid response to the motion of small particles driven by external forces. Correlations in the particle Brownian motion in dilute solutions can be studied. In the latter case the time of observation should be larger than the corresponding viscous relaxation time. In both cases the particle motion is required to be sufficiently slow to ensure the smallness of the force acting on the fluid. In the case of Brownian motion, this condition is satisfied due to the smallness of the Langevin forces driving the motion.

The structure of the paper is as follows. In Sec. II we consider the general hydrodynamic equations and the boundary conditions associated with membranes. Section III is dedicated to the fluid confined by two parallel solid or membrane walls. The equations for the flow in spherical geometry systems (a nearly spherical vesicle; see below) are solved in Sec. IV. The solution is given as the sum of spherical harmonics. Application of the results to Brownian motion correlations is presented in Sec. V. Discussion of the results is given in Sec. VI.

II. BASIC RELATIONS

A fluid flow is characterized by a velocity field $\boldsymbol{v}$. For relatively slow flows, when sound excitation can be neglected, and at small Reynolds numbers, when nonlinearity of the flow is vanishingly small, the flow is governed by the Stokes equation

$$\varrho \partial_t \boldsymbol{v} = \eta \nabla^2 \boldsymbol{v} - \nabla p + \boldsymbol{f}, \qquad (2.1)$$

where ϱ is the fluid mass density, η is its dynamic viscosity, p is the pressure, and $\boldsymbol{f}$ is the force density. It is determined by Langevin forces (thermal noise) and by external forces applied to the fluid. In the above conditions ϱ can be treated as constant and the incompressibility condition $\nabla \boldsymbol{v} = \boldsymbol{0}$ is satisfied. The incompressibility leads to the equation

$$\nabla^2 p = \nabla \boldsymbol{f} \qquad (2.2)$$

for the pressure. Relation (2.2) implies that the pressure is excluded from the set of dynamic variables; it has to be considered as an auxiliary field ensuring fluid incompressibility.

We investigate the fluid reaction to a stationary force $\boldsymbol{F}$ applied at the point $\boldsymbol{r}_0$. The solution can then be used to find the response to an arbitrarily distributed force. We should solve the equation

$$\eta \nabla^2 \boldsymbol{v} - \nabla p + \boldsymbol{F} \delta(\boldsymbol{r} - \boldsymbol{r}_0) = \boldsymbol{0}. \qquad (2.3)$$

For an unbounded fluid, the solution of Eq. (2.3) is [6]

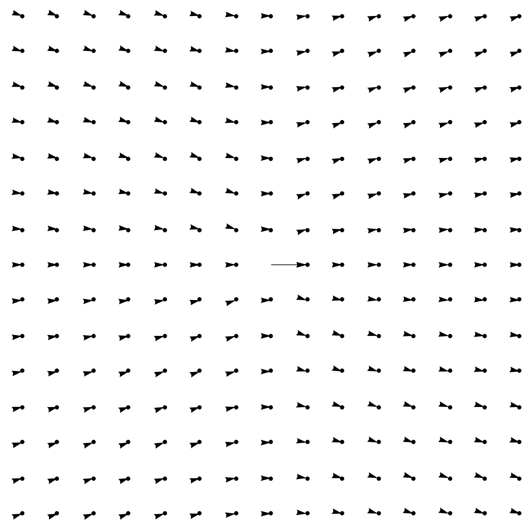

FIG. 1. Axial cross section of the velocity field (2.4) for the unbounded fluid.

$$\boldsymbol{v}^{(0)} = \frac{1}{8\pi\eta}\left(\frac{\boldsymbol{F}}{x} + \frac{(\boldsymbol{F}\boldsymbol{x})\boldsymbol{x}}{x^3}\right), \tag{2.4}$$

where $\boldsymbol{x}=\boldsymbol{r}-\boldsymbol{r}_0$ (see Fig. 1). Below, we analyze more sophisticated cases, taking into account the presence of solid boundaries and membranes. The expression (2.4) remains valid near the point where the force is applied, whereas it is essentially modified on distances of the order of and larger than the distance from the point $\boldsymbol{r}_0$ to boundaries and/or membranes.

It is convenient to represent the solution of the problem as

$$\boldsymbol{v} = \boldsymbol{v}^{(0)} + \boldsymbol{u}, \tag{2.5}$$

where $\boldsymbol{v}^{(0)}$ is the velocity field (2.4) induced in the unbounded fluid. The contribution $\boldsymbol{u}$ satisfies the homogeneous equation

$$\eta\nabla^2\boldsymbol{u} - \nabla p = \boldsymbol{0}, \tag{2.6}$$

which has to be supplemented by the boundary condition $\boldsymbol{u}=-\boldsymbol{v}^{(0)}$ at solid walls. The boundary conditions in the presence of membranes are more complicated and require special consideration.

A. Membranes

When we discuss membranes, we have in mind lipid bilayers. The physical properties of such objects have been extensively studied, both experimentally and theoretically (see e.g., the books [7–9] and the reviews [10–12]). In our investigation we treat membranes as infinitely thin films i.e., as two-dimensional objects. This is justified provided the characteristic scale of a problem is much larger than the membrane thickness, a condition which is assumed to be satisfied.

The membrane energy is related to its elasticity and shape deformation. The former is described in terms of the membrane surface tension σ and the latter depends on the membrane curvature. We exploit the main contribution to the curvature energy, which can be written as the following surface integral [13–15]:

$$\mathcal{F} = \frac{\kappa}{2}\int dS\, H^2. \tag{2.7}$$

Here H is the membrane mean curvature, $H=1/R_1+1/R_2$, R_1 and R_2 being the local curvature radii of the membrane, and κ is the bending modulus also known as the Helfrich modulus. Note that the mean curvature can be written as $H=\partial_i l_i$ where $\boldsymbol{l}$ is a unit vector perpendicular to the membrane. To calculate H the field $\boldsymbol{l}$ can be arbitrary extended into the third direction (away from the membrane) since (due to the condition $l^2=1$) $\partial_i l_i \equiv \partial_i^\perp l_i$, where $\partial_i^\perp = \partial_i - l_i l_k \partial_k$ is a special derivative along the membrane.

In hydrodynamics, one has to take into account fluids at both sides of the membrane. In order to solve the problem of fluid motion around the membrane, one should formulate the boundary conditions on it. We neglect the inertial effects; then the membrane moves with the velocity of the surrounding fluid and the velocity field $\boldsymbol{v}$ is continuous on the membrane. Next, we assume that the membrane is incompressible, a condition that can be written as $\partial_i^\perp v_i=0$. Note that due to the three-dimensional (3D) incompressibility $\partial_i v_i=0$ of the flow, the membrane incompressibility means that the condition $l_i l_j \partial_i v_j=0$ must be satisfied at both sides of the membrane. In addition, force balance should be satisfied on the membrane,

$$\phi_i = \lfloor p \rfloor l_i - l_k \lfloor \eta(\partial_k v_i + \partial_i v_k) \rfloor, \tag{2.8}$$

where ϕ_i is the membrane force (per unit area) and the floor brackets designate a jump of the corresponding quantity on the membrane. The right-hand side of Eq. (2.8) represents the momentum flux to the membrane related to pressure and the viscous stress tensor. Projecting relation (2.8), we obtain

$$l_i\phi_i = \lfloor p \rfloor, \quad \delta_{ij}^\perp \phi_j = -\delta_{ij}^\perp l_k \lfloor \eta(\partial_j v_k + \partial_k v_j) \rfloor \tag{2.9}$$

for the normal and tangential components of the force. Here $\delta_{ik}^\perp = \delta_{ik} - l_i l_k$ is the projector to the membrane.

The membrane force (per unit area) can be written as $\phi_i = -\partial_k^\perp T_{ik}^{(s)}$, where $T_{ik}^{(s)}$ is the membrane stress tensor. There are three contributions to the stress tensor, related to the membrane surface tension, the membrane curvature, and the membrane viscosity:

$$T_{ik}^{(s)} = -\sigma\delta_{ik}^\perp + T_{ik}^{(\kappa)} - \zeta\delta_{ij}^\perp\delta_{kn}^\perp(\partial_j v_n + \partial_n v_j), \tag{2.10}$$

where σ is the membrane surface tension and ζ is the membrane 2D shear viscosity coefficient. An explicit expression for the bending contribution is

$$T_{ik}^{(\kappa)} = \kappa\left(-\frac{1}{2}H^2\delta_{ik}^\perp + H\partial_i^\perp l_k - l_i\partial_k^\perp H\right); \tag{2.11}$$

it was derived in [16] (see also [17]). Calculating the derivative of $T_{ik}^{(s)}$, one finds from Eqs. (2.10) and (2.11) that $\phi_i = \phi_i^{(\kappa)} + \phi_i^{(\sigma)} + \phi_i^{\zeta}$, where

$$\phi_i^{(\kappa)} = \kappa[H(H^2/2 - 2K) + \Delta^\perp H] l_i, \tag{2.12}$$

$$\phi_i^{(\sigma)} = -H\sigma l_i + \partial_i^\perp \sigma, \tag{2.13}$$

$$\phi_i^{(\zeta)} = \zeta[\delta_{ij}^{\perp}\Delta^{\perp}v_j - Hl_n\partial_i^{\perp}v_n - -2l_i(\partial_n^{\perp}l_j)\partial_j^{\perp}v_n], \tag{2.14}$$

which are the curvature, tension, and viscous forces, respectively. Here $K=(R_1R_2)^{-1}=[(\partial_i l_i)^2-\partial_k l_i\partial_i l_k]/2$ is the Gaussian curvature and $\Delta^{\perp}=\partial_i^{\perp}\partial_i^{\perp}$ is a Beltrami-Laplace operator. Note that the force (2.12) can be obtained as a coefficient of the bending energy (2.7) variation at an infinitesimal membrane deformation [18]. The force (2.13) was discussed in Ref. [19].

Now, we can write the boundary conditions (2.9) explicitly:

$$\kappa[H(H^2/2-2K)+\Delta^{\perp}H] - H\sigma - 2\zeta(\partial_n^{\perp}l_j)\partial_j^{\perp}v_n = \lfloor p \rfloor, \tag{2.15}$$

$$\partial_i^{\perp}\sigma + \zeta(\delta_{ij}^{\perp}\Delta^{\perp}v_j - Hl_n\partial_i^{\perp}v_n) = -\delta_{ij}^{\perp}l_k\lfloor\eta(\partial_j v_k + \partial_k v_j)\rfloor. \tag{2.16}$$

In order to get rid of σ in the last equation, we apply the $l_j\epsilon_{jki}\partial_k^{\perp}=l_j\epsilon_{jki}\partial_k$ operator to both sides and obtain

$$\zeta l_j\epsilon_{jmi}[\partial_m^{\perp}\Delta^{\perp}v_i - \partial_m^{\perp}(Hl_n)\partial_i^{\perp}v_n] + l_j\epsilon_{jmi}(\partial_m^{\perp}l_k)\lfloor\eta(\partial_i v_k + \partial_k v_i)\rfloor + l_j l_k\epsilon_{jmi}\partial_m\lfloor\eta\partial_k v_i\rfloor = 0. \tag{2.17}$$

Note that this condition does not contain the membrane rigidity described by the energy (2.7). However, it is sensitive to the membrane viscosity ζ.

If an infinitesimally small force $\boldsymbol{F}$ is applied to a fluid where the membranes have equilibrium shapes, then one can regard the shapes as unchanged when applying the boundary conditions, since the membrane deformation is infinitesimally small, too. The fact that the boundary conditions should be applied at the exact interface position represents an effect of higher order (see also [4]). Then looking for the response to the force we come to a problem formulated for fixed membrane positions. In terms of the velocity field, it is written as $l_i v_i=0$, meaning that the velocity perpendicular to the membrane is equal to zero, which also reflects the fact that we consider stationary solutions. Let us stress that the longitudinal velocity is generally nonzero at the membrane. Thus the boundary conditions for the velocity field at the membrane are $l_i v_i=0$, the membrane incompressibility $\partial_i^{\perp}v_i=0$, and Eq. (2.17). These three conditions substitute the no-slip condition $\boldsymbol{v}=\boldsymbol{0}$ valid for solid walls.

We also neglect thermal fluctuations of the membranes. This is justified by the small value of the ratio $T/(2\pi\kappa)$ (where T is the temperature), which is usually of order 10^{-2}.

III. FLAT GEOMETRY

In this section we examine a fluid's response to a point force in the presence of nearly flat membranes. We assume that there are some parallel nearly flat membranes, which corresponds, say, to a dilute lyotropic phase. For methodical reasons, we first present results for the fluid response to a point force in a flat capillary, a problem that was previously considered using a different method in Ref. [3]. We reproduce the result of this paper in a different way in order to develop a method that we then use for the membrane system. Then we pass to the case of two parallel membranes.

A. Flat capillary

Here, we consider the fluid response to a point force $\boldsymbol{F}$ in a flat capillary where the fluid is confined between two parallel walls. The walls are assumed to be separated by a distance h and the force is applied at a distance w from one of the walls. We choose a reference system where the Z axis is perpendicular to the walls and the X-Y plane coincides with one of them. The second wall is determined by the condition $z=h$. The coordinates of the force application point can be set to be $\boldsymbol{r}_0=(0,0,w)$ by an appropriate shift of the origin. Then the problem is formulated as the solution of Eq. (2.3) supplemented by the boundary conditions $\boldsymbol{v}=\boldsymbol{0}$ on the walls i.e., at $z=0$ and at $z=h$. This problem was solved in [3].

To find $\boldsymbol{u}$ [see (2.5)], it is convenient to produce a partial Fourier transformation

$$\boldsymbol{u}(\rho_\alpha,z) = \int \boldsymbol{u}(k_\alpha,z)\exp(ik_\alpha\rho_\alpha)\frac{d^2k}{(2\pi)^2}, \tag{3.1}$$

and similarly for the pressure p. Here greek indices denote the X and Y components and $\boldsymbol{\rho}=(x,y)$ is a two-dimensional radius vector. In terms of the Fourier transforms, the incompressibility condition and Eq. (2.6) can be rewritten as follows:

$$ik_\alpha u_\alpha + \partial_z u_z = 0, \tag{3.2}$$

$$-\eta k^2 u_\alpha + \eta\partial_z^2 u_\alpha - ik_\alpha p = 0, \tag{3.3}$$

$$-\eta k^2 u_z + \eta\partial_z^2 u_z - \partial_z p = 0. \tag{3.4}$$

In order to formulate the boundary conditions for Eqs. (3.2)–(3.4) we need the Fourier transform of the velocity (2.4),

$$v_z^{(0)}(z,\boldsymbol{k}) = \frac{e^{-|z-w|k}}{\eta}\left(\frac{F_z}{4k} + \frac{F_z|z-w|}{4} - \frac{iF_\beta k_\beta(z-w)}{4k}\right), \tag{3.5}$$

$$v_\alpha^{(0)}(z,\boldsymbol{k}) = \frac{1}{\eta}\left(\frac{F_\alpha e^{-|z-w|k}}{2k} - \frac{k_\alpha F_\beta k_\beta|z-w|e^{-|z-w|k}}{4k^2} - \frac{k_\alpha F_\beta k_\beta e^{-|z-w|k}}{4k^3} - \frac{k_\alpha F_z i(z-w)e^{-|z-w|k}}{4k}\right). \tag{3.6}$$

Equations (3.2)–(3.4) are ordinary differential equations with constant coefficients. Their solution corresponding to the boundary conditions $\boldsymbol{u}(0)=-\boldsymbol{v}^{(0)}(0)$ and $\boldsymbol{u}(h)=-\boldsymbol{v}^{(0)}(h)$ can be found explicitly. Then $\boldsymbol{u}(x,y,z)$ can be written as a Fourier integral. The final expression is quite cumbersome, and can be found in Appendix A. However, if one is interested in the velocity field at large distances from the force application point, $\rho\gg h$, one can find the asymptotic form of $\boldsymbol{v}$. Expanding the Fourier components of the velocity in powers of kh and leaving the lowest nonvanishing terms, one finds

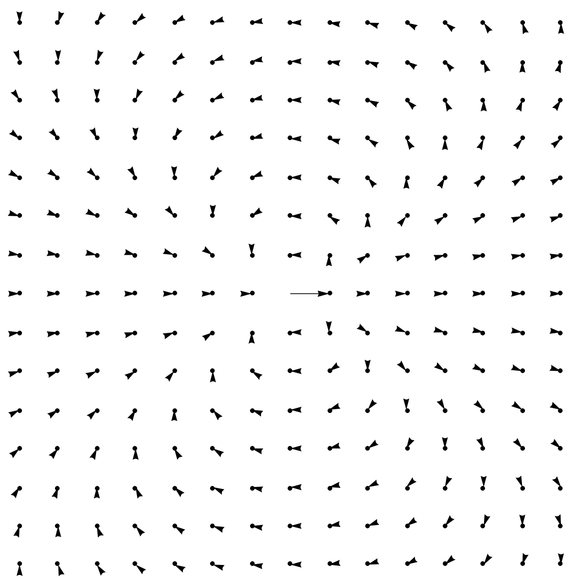

FIG. 2. Velocity field (3.8) between two solid walls for $z=h/2$, $w=h/2$.

$$v_\alpha = \frac{3(h-w)(z-h)wz}{\eta h^3}\frac{F_\beta k_\beta k_\alpha}{k^2}. \tag{3.7}$$

Performing the Fourier transformation (3.1) of this expression, we obtain (see Fig. 2)

$$v_\alpha \approx \frac{3(h-w)(z-h)wz}{2\pi\eta h^3}\left(\frac{F_\alpha}{\rho^2} - \frac{2(F_\beta\rho_\beta)\rho_\alpha}{\rho^4}\right), \tag{3.8}$$

in accordance with the result obtained in [3]. Note that the Z component of velocity is of higher order in h/ρ than the X,Y components.

B. Nearly flat membranes

Here we solve a problem similar to the one in Sec. III A with the only difference that now we consider a fluid that is bounded by membranes instead of solid walls. We recall that we neglect thermal fluctuations and deformations of the membranes when applying boundary conditions.

We assume that two membranes are perpendicular to the Z axis and have coordinates $z=0$ and $z=h$. The point force $\boldsymbol{F}$ is assumed to be applied at the point $(0,0,w)$ where $0<w<h$. The membrane incompressibility is written as $\partial_x v_x + \partial_y v_y = 0$, which means that $\partial_z v_z = 0$ must be satisfied at both sides of the membrane (as a consequence of the fluid incompressibility). The membrane immobility is written as $v_z=0$ at the membrane. The boundary condition (2.17) can be rewritten as

$$\zeta(\partial_x^2 + \partial_y^2)(\mathrm{curl}\ \boldsymbol{v})_z + \eta\lfloor\partial_z(\mathrm{curl}\ \boldsymbol{v})_z\rfloor = 0, \tag{3.9}$$

where the fluids at both sides of the membrane are assumed to be the same. This set of conditions is sufficient to determine the velocity field completely. We recall that the velocity field between the membranes is decomposed as written in (2.5).

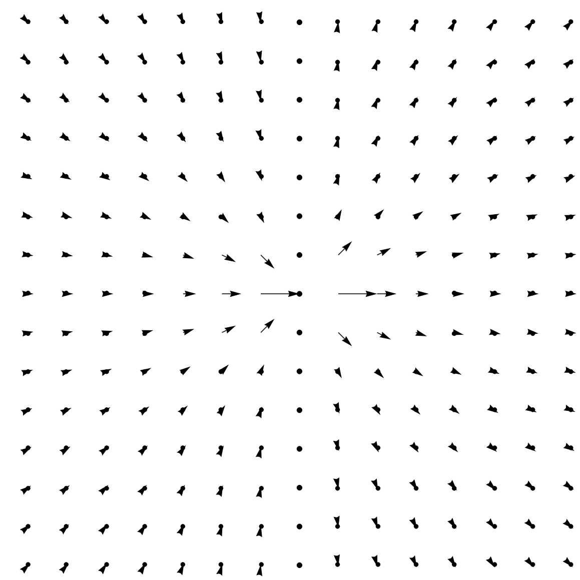

FIG. 3. Velocity field (3.11) between the membranes.

As previously, we pass to the partial Fourier transform and solve Eqs. (3.2)–(3.4). However, now we should solve these equations for three regions: $z<0$, $0<z<h$, and $z>h$. The boundary conditions $v_z=0$ and $\partial_z v_z=0$ correspond to the flows in all three regions, while the condition (3.9) relates the solutions on different sides of the membranes. They should be supplemented with the condition of zero velocity as $z\to\pm\infty$. The equations can be solved explicitly and then $\boldsymbol{v}(\boldsymbol{\rho},z)$ is written as a 2D Fourier integral. As above, the expressions are cumbersome; we do not present them here (they can be found in Appendix A).

Let us first consider the expression for the velocity between the membranes. Again, we are interested in the asymptotic behavior at large distances $\rho\gg h$ or $kh\ll 1$. The main term of the expansion in kh is

$$u_\alpha = \frac{1}{2\eta}\left(\frac{F_\alpha}{k} - \frac{F_\beta k_\beta k_\alpha}{k^3}\right). \tag{3.10}$$

Here, we have neglected the internal membrane viscosity ζ. The approximation is correct provided $\rho\gg\zeta/\eta$. After the Fourier transformation we find (see Fig. 3):

$$v_\alpha \approx \frac{1}{4\eta\pi}\frac{\rho_\alpha F_\beta\rho_\beta}{\rho^3}. \tag{3.11}$$

We see that the velocity of the fluid confined between the membranes decays with the distance ρ more slowly than for the solid wall case. The pressure between the membranes is the same as between the solid walls.

The flow outside the membranes is parallel to them. It can be written explicitly (in Fourier representation) as

$$u_\alpha^{(\mathrm{out})} = \frac{e^{-\gamma k}}{2\eta}\left(\frac{F_\alpha}{k} - \frac{F_\beta k_\beta k_\alpha}{k^3}\right), \tag{3.12}$$

where $\gamma = |z-w|$. Again, we have neglected the effects of the internal membrane viscosity ζ. We see that the velocity depends only on the distance from the force application point, but not on the position of the membrane (as long as there is a membrane between the point of observation and the point of the force application). We can perform a Fourier transformation with the expression (3.12), which gives

$$u_\alpha^{(\mathrm{out})} = \frac{1}{4\eta\pi}\left(\frac{F_\alpha}{\rho_\star} + \frac{\gamma F_\alpha}{\rho^2} - \frac{F_\alpha \rho_\star}{\rho^2} - \frac{2\gamma F_\beta \rho_\beta \rho_\alpha}{\rho^4} + \frac{F_\beta \rho_\beta \rho_\alpha}{\rho^2 \rho_\star} + \frac{2\gamma^2 F_\beta \rho_\beta \rho_\alpha}{\rho^4 \rho_\star}\right), \tag{3.13}$$

where $\rho_\star = \sqrt{\rho^2 + \gamma^2}$. The pressure outside the membranes is not affected by the force $\boldsymbol{F}$.

It can be shown that this velocity field remains the same if we add more parallel membranes to the system. In other words, if one has a system of multiple parallel membranes (lamellar lyotropic phase), then the velocity induced by a point force will be given by the formula (3.13) as long as this point is separated from the application point by at least one membrane.

We derived the expression (3.12) and hence (3.13) by neglecting the membrane internal viscosity. Its effect at large distances ($\rho \gg \zeta/\eta$) is of higher order in γ/ρ than the expressions that were obtained previously. Let us now consider small distances $\rho \ll \zeta/\eta$. The expression (3.12) should be substituted with the following one:

$$u_\alpha^{(\mathrm{out})} = \frac{e^{-\gamma k}}{2(\eta + \zeta k/2)}\left(\frac{F_\alpha}{k} - \frac{F_\beta k_\beta k_\alpha}{k^3}\right). \tag{3.14}$$

One can see that it differs from (3.12) only by the denominator: η is replaced by $\eta + \zeta k/2$. For small ρ we can neglect the fluid viscosity term, and simultaneously cut down the Fourier integrals on the lower limit to get rid of the divergency. As a result we get the following answers for two different cases:

$$u_\alpha^{(\mathrm{out})} \approx \frac{F_\alpha}{4\pi\zeta}\ln\frac{\zeta}{\eta\gamma}, \quad \gamma \gg \rho, \tag{3.15}$$

$$u_\alpha^{(\mathrm{out})} \approx \frac{F_\alpha}{4\pi\zeta}\ln\frac{\zeta}{\eta\rho}, \quad \gamma \ll \rho. \tag{3.16}$$

We see that in this case the behavior of the velocity is logarithmic, not powerlike as for large distances.

C. Summary

It is worth making some conclusions and compare the results obtained above. We were mostly concerned with the long-distance behavior of the induced velocity. We saw that the presence of solid walls produces a damping effect on the flow: the velocity field between two solid walls decays with the distance faster than in an unbounded liquid. Note that both the normal force and velocity are damped more strongly than the tangential one, which is expressed in the fact that v_z decays faster than v_α and the contribution of F_z to the flow is negligible compared to F_α.

If we compare the solid wall with the membrane case, we first note that they both damp the normal force and velocity. Indeed, as we saw before, the normal components of velocity are identical for both cases. We will later see (for spherical geometry) that this effect is quite general. The tangential flow is, on the contrary, not affected by the membranes so significantly as by solid walls. Comparing expressions (2.4) and (3.11), we see that the membranes do not change the decay law of induced velocity, compared to an unbounded fluid, but change its direction (see Figs. 1–3).

IV. SPHERICAL GEOMETRY

We now turn to the case of closed membranes which are usually called vesicles. Since both fluid and membrane are considered to be incompressible, then a vesicle can be described by its volume V and area S. It is convenient to write the area as $S = (4\pi + \Delta)R^2$, where R is the radius, corresponding to the sphere with volume $V = \frac{4}{3}\pi R^3$. We consider the simplest case of a nearly spherical vesicle, which means that $\Delta \ll 1$. This allows us to apply the boundary conditions directly on the surface of the sphere, neglecting the membrane deformation and its deviation from a spherical shape. The point force is assumed to be applied to the fluid inside the vesicle; we need to find the velocity field both inside and outside it. We use the same method of representation of the solution as the sum of two parts as in (2.5). So, as before, we have to solve the homogeneous Stokes equation with some boundary conditions. We expand the solution in terms of spherical functions. According to Lamb [20] (see also [6]), a general solution of the Stokes equation can be represented as

$$\boldsymbol{u} = \sum_{l=0}^{\infty}\left(\mathrm{curl}(\boldsymbol{r}\chi_l) + \nabla\Phi_l + \frac{(l+3)}{2\eta(l+1)(2l+3)}r^2\,\nabla\, p_l - \frac{l}{\eta(l+1)(2l+3)}\boldsymbol{r}p_l\right) \tag{4.1}$$

for an internal problem, and as

$$\boldsymbol{u} = \sum_{l=0}^{\infty}\left(\mathrm{curl}(\boldsymbol{r}\tilde{\chi}_l) + \nabla\tilde{\Phi}_l + \frac{(l-2)}{2\eta l(1-2l)}r^2\,\nabla\,\tilde{p}_l - \frac{l+1}{\eta l(1-2l)}\boldsymbol{r}\tilde{p}_l\right) \tag{4.2}$$

for an external problem. Here χ_l, Φ_l, p_l and $\tilde{\chi}_l, \tilde{\Phi}_l, \tilde{p}_l$ are harmonic functions,

$$\chi_l = r^l \sum_m \chi_{lm} Y_{lm}(\theta,\varphi),$$

$$\tilde{\chi}_l = r^{-l-1} \sum_m \tilde{\chi}_{lm} Y_{lm}(\theta,\varphi), \tag{4.3}$$

and analogously for Φ_l, p_l and $\tilde{\Phi}_l, \tilde{p}_l$. Note that p_l or $\tilde{p}_l$ represents the expansion of the pressure.

A. Spherical vessel

As in the previous section we first solve the Stokes equation for the solid wall case. We consider a fluid in a spherical vessel with solid walls and examine a flow induced by the point force $\boldsymbol{F}$ applied inside the vessel. Then the expansion (4.1) can be used to describe the flow. One can check that the boundary condition $\boldsymbol{v}=\boldsymbol{0}$ on the solid wall means that for a spherical vessel the relations

$$u_r = - v_r^{(0)}, \tag{4.4}$$

$$r\partial_r u_r = - r\partial_r v_r^{(0)}, \tag{4.5}$$

$$\boldsymbol{r} \cdot \text{curl } \boldsymbol{u} = - \boldsymbol{r} \cdot \text{curl } \boldsymbol{v}^{(0)} \tag{4.6}$$

must be satisfied at the vessel walls, that is, at $r=R$, where R is the vessel radius. The boundary conditions (4.4)–(4.6) can be rewritten in terms of the functions χ_l, Φ_l, p_l since at $r=R$

$$u_r = \sum_l \left(\frac{lR}{2\eta(2l+3)} p_l + \frac{l}{R}\Phi_l \right), \tag{4.7}$$

$$r\partial_r u_r = \sum_l \left(\frac{l(l+1)R}{2\eta(2l+3)} p_l + \frac{l(l-1)}{R}\Phi_l \right), \tag{4.8}$$

$$\boldsymbol{r} \cdot \text{curl } \boldsymbol{u} = \sum_l l(l+1)\chi_l. \tag{4.9}$$

Let us choose the spherical coordinates with the polar axis directed along the line connecting the center of the sphere and the force application point. Let a denote the distance between them, and θ_0 the angle between the polar axis and the force vector. Longitude will be counted from the force vector.

To use the relations (4.4)–(4.6) one has to expand the boundary values of $\boldsymbol{v}^{(0)}$ in terms of surface spherical functions,

$$v_r^{(0)} = \sum X_l, \quad r\partial_r v_r^{(0)} = \sum Y_l, \quad (\text{curl } \boldsymbol{v}^{(0)})_r = \sum Z_l, \tag{4.10}$$

at $r=R$. Explicit expressions for these functions can be found in Appendix B. Using relations (4.4)–(4.6) we can express the functions χ_l, Φ_l, and p_l in terms of X_l, Y_l, and Z_l:

$$\Phi_l = \frac{R}{2l}\left(\frac{r}{R}\right)^l [Y_l - (l+1)X_l], \tag{4.11}$$

$$p_l = \frac{\eta(2l+3)}{lR}\left(\frac{r}{R}\right)^l [(l-1)X_l - Y_l], \tag{4.12}$$

$$\chi_l = -\frac{1}{l(l+1)}\left(\frac{r}{R}\right)^l Z_l. \tag{4.13}$$

Substituting these expressions into Eq. (4.1) we find $\boldsymbol{u}$ and then the velocity $\boldsymbol{v}$ inside the vessel in accordance with Eq. (2.4).

B. Nearly spherical vesicle

Let us now turn to the vesicle. We consider the general case when the fluids inside and outside the vesicle are different. Let η_1 and η_2 denote the fluid viscosities inside and outside the vesicle, respectively. If the force $\boldsymbol{F}$ is applied inside the vesicle, the latter has to be moving as a whole with the Stokes velocity $\boldsymbol{F}/(6\pi\eta_2 R)$ where R is the vesicle radius (see, for example, [5]). The term has to be added to the expression (4.1) to ensure the right momentum flux outside the vesicle.

Let us formulate the boundary conditions, required to solve the Stokes equation. As previously, we neglect the membrane shape perturbations and consider the membrane to be incompressible. For the nearly spherical vesicle we obtain

$$\partial_r v_r = 0, \quad v_r = \frac{(\boldsymbol{F},\boldsymbol{n})}{6\pi\eta_2 R} = \frac{F(\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos\varphi)}{6\pi\eta_2 R}, \tag{4.14}$$

where the first condition represents the membrane incompressibility and the second condition means that the vesicle moves with the Stokes velocity $\boldsymbol{F}/(6\pi\eta_2 R)$ ($\boldsymbol{n}$ denotes the unit vector in the direction of $\boldsymbol{r}$). Next, we obtain from Eq. (2.17)

$$\lfloor \eta\partial_r(\text{curl } \boldsymbol{v})_r \rfloor = -\zeta\partial_r^2(\text{curl } \boldsymbol{v})_r - 4\zeta r^{-1}\partial_r(\text{curl } \boldsymbol{v})_r. \tag{4.15}$$

The right-hand side of this expression contains the velocity on the membrane. One can use either the internal or external fluid velocity since they coincide on the membrane. The last boundary condition is

$$\lfloor(\text{curl } \boldsymbol{v})_r\rfloor = 0, \tag{4.16}$$

which is simply a consequence of a tangential velocity continuity.

Let us now find the flow caused by the point force acting inside the vesicle. Equation (4.5) remains the same and Eq. (4.4) should be replaced by the following one:

$$u_r = -v_r^{(0)} + \frac{(\boldsymbol{F},\boldsymbol{n})}{6\pi\eta_2 R}, \tag{4.17}$$

in accordance with (4.14). Together with (4.15) and (4.16), these equations constitute the full set of boundary conditions for the nearly spherical vesicle enabling one to find the velocity field. We first find p_l and Φ_l. We conclude that these functions are the same for the vesicle and for the solid wall case in the inner region, apart from the addition to Φ of

$$\Phi_0 = \frac{(\boldsymbol{F},\boldsymbol{r})}{6\pi\eta_2 R}, \tag{4.18}$$

which corresponds to the fluid inside the vesicle moving as a whole with the velocity $\boldsymbol{F}/(6\pi\eta_2 R)$. In the outer region all Φ_l and p_l are zero except for the terms

$$\Phi_1 = \frac{R^2}{24\pi\eta_2}\frac{(\boldsymbol{F},\boldsymbol{r})}{r^3}, \tag{4.19}$$

$$p_1 = \frac{(\boldsymbol{F},\boldsymbol{r})}{4\pi r^3}. \tag{4.20}$$

These terms produce the flow outside the vesicle, which is the same as if the vesicle were a solid sphere moving through the fluid,

$$\boldsymbol{v} = \frac{1}{8\pi\eta_2}\frac{\boldsymbol{F} + (\boldsymbol{F},\boldsymbol{n})\boldsymbol{n}}{r} + \frac{R^2}{24\pi\eta_2}\frac{\boldsymbol{F} - 3(\boldsymbol{F},\boldsymbol{n})\boldsymbol{n}}{r^3}. \tag{4.21}$$

This result was expected since the boundary conditions corresponding to it are the same for the membrane and the solid sphere.

It only remains to find χ_l inside and outside the vesicle. The boundary conditions lead to a set of two equations, which can be found in Appendix B. Solving them, we find

$$\chi_l^{\text{out}} = \frac{F\sin\theta_0}{4\pi}\frac{(2l+1)}{l(l+1)N_l}\frac{a^l}{r^{l+1}}P_l^1 \sin\varphi, \tag{4.22}$$

$$\chi_l^{\text{in}} = -\frac{F\sin\theta_0(l+2)}{4\pi l(l+1)}\frac{[\eta_2 - \eta_1 + (\zeta/R)(l-1)]}{N_l}\frac{a^l r^l}{R^{2l+1}}P_l^1 \sin\varphi, \tag{4.23}$$

where P_l are Legandre polynomials of order l, P_l^1 are their associated polynomials of the first order, and $N_l = [\eta_1(l-1) + \eta_2(l+2) + (\zeta/R)(l-1)(l+2)]$. We see that the internal membrane viscosity does not affect the first harmonic. The corresponding contribution to the velocity is the most slowly decaying one at large distances from the vesicle. It is

$$\chi_1^{\text{out}} = \frac{Fa\sin\theta_0\sin\theta\sin\varphi}{8\pi\eta_2 r^2}, \tag{4.24}$$

or in vector notation

$$\chi_1^{\text{out}} = -\frac{(\boldsymbol{r}\cdot[\boldsymbol{F}\times\boldsymbol{a}])}{8\pi\eta_2 r^3}. \tag{4.25}$$

This expression does not contain the viscosity of the fluid inside the vesicle. Finally we have

$$\boldsymbol{v} = \text{curl}(\boldsymbol{r}\chi) = \frac{1}{8\pi\eta_2}\frac{[\boldsymbol{r}\times[\boldsymbol{F}\times\boldsymbol{a}]]}{r^3} = \frac{R^3}{r^3}[\boldsymbol{\Omega}\times\boldsymbol{r}],$$

$$\boldsymbol{\Omega} = \frac{[\boldsymbol{a}\times\boldsymbol{F}]}{8\pi\eta_2 R^3}, \tag{4.26}$$

the expected result. The surface of the vesicle is rotating as a whole, so the internal viscosity does not play a role, and the flow outside is the same as for a solid sphere rotating with the angular velocity Ω.

Note that, if we look at the limit $\zeta\to\infty$, we find that all harmonics except for the first one vanish, while the latter remains the same [together with expression (4.26)] as is seen from (4.22). The translational part (4.21) also does not change. This result could have been predicted, since the above limit corresponds to a solid weightless spherical shell which can only move and rotate.

As was shown before, fluid flow both inside and outside the vesicle can be represented as a sum of two contributions. The first one is due to the motion of the vesicle as a whole and is given by expression (4.21). The second one represents "rotational" motion and corresponds to (4.22) combined with (4.2) and can be written as a sum of spherical harmonics. At large distances $r\gg a$ the former is stronger, since it decays more slowly, i.e., as $1/r$.

C. Summary

We now can compare the results obtained for the plane and spherical geometries. From expression (4.22) one can see that if the force is acting in the z direction (normal to the spherical surface) all the spherical harmonics vanish (except for the one responsible for translation). This situation corresponds to the force acting in the normal direction for the plane membrane case. We also note that (again, apart from the translational part) the flow outside the vesicle does not have a radial part. In other words, there is only a tangential (to the membrane surface) flow, again as for the plane membranes. There is the further analogy that the pressure inside the vesicle is the same as in a spherical vessel, and the pressure outside is constant.

V. BROWNIAN MOTION

Having solved the hydrodynamic equations, we now analyze the results obtained. In this section we will show how our solutions can be applied to the investigation of the particle displacement correlations in their Brownian motion. This is achieved by applying the fluctuation-dissipation theorem to a system of particles immersed in a fluid.

A. Fluctuation-dissipation theorem

The hydrodynamic interaction of particles immersed in a fluid can be tested by investigating their Brownian motion. Let $\boldsymbol{X}_a$ be the positions of the particles (enumerated by the index a). The Brownian motion can be characterized in terms of the correlation functions $\langle X_{a,i}(t)X_{b,j}(0)\rangle$ where the angular brackets designate averaging over the realizations and the indices i and j designate components of the vectors. Experimentally, it is convenient to measure the correlation functions $\langle [X_{a,i}(t)-X_{a,i}(0)][X_{b,j}(t)-X_{b,j}(0)]\rangle$, expressed in terms of the particle displacements from their initial positions.

According to the fluctuation-dissipation theorem (see, e.g., Ref. [21]),

$$\langle X_{a,i}X_{b,j}\rangle_\omega = \frac{2T}{\omega}\mathrm{Im}\ \alpha_{ab,ij}(\omega), \quad (5.1)$$

where T is the fluid temperature,

$$\langle X_{a,i}(t)X_{b,j}(0)\rangle = \int \frac{d\omega}{2\pi}e^{-i\omega t}\langle X_{a,i}X_{b,j}\rangle_\omega, \quad (5.2)$$

and $\alpha_{ab,ij}(\omega)$ is the linear susceptibility of the system. It determines the displacement of particle a in the i direction provided the external force $\boldsymbol{F}_{b,j}(t)$ is applied to particle b. In the Fourier representation

$$\langle X_{a,i}\rangle_\omega = \sum_{b,j}\alpha_{ab,ij}(\omega)F_{b,j}(\omega). \quad (5.3)$$

For the particle velocities $\boldsymbol{V}_a$, this relation (5.3) can be rewritten as follows:

$$\langle V_{a,i}\rangle_\omega = \sum_{b,j}\beta_{ab,ij}(\omega)F_{b,j}(\omega), \quad (5.4)$$

where $\beta=-i\omega\alpha$. Then it follows from the relation (5.1) that

$$\langle[X_{a,i}(t)-X_{a,i}(0)][X_{b,j}(t)-X_{b,j}(0)]\rangle = \int\frac{d\omega}{2\pi}[1-\cos(\omega t)]\frac{4T}{\omega^2}\mathrm{Re}\ \beta_{ab,ij}(\omega). \quad (5.5)$$

Under the action of stationary forces, the particles move with constant velocities in the fluid. Therefore in the limit $\omega\to 0$ the matrix $\beta_{ab,ij}(\omega)$ tends to a constant, whereas the matrix $\alpha_{ab,ij}(\omega)$ has a simple pole at $\omega=0$. The susceptibility $\beta_{ab,ij}(\omega)$ remains approximately constant for frequencies less than $\eta/(\varrho L^2)$ where L is the interparticle distance. Therefore if t is much larger than $\varrho L^2/\eta$ then $\beta_{ab,ij}(\omega)$ can be substituted by $\beta_{ab,ij}(\omega=0)$ in Eq. (5.5). Then, calculating the integral over ω, one finds

$$\langle[X_{a,i}(t)-X_{a,i}(0)][X_{b,j}(t)-X_{b,j}(0)]\rangle = 2Tt\beta_{ab,ij}(\omega=0), \quad (5.6)$$

where we have taken into account that $\beta_{ab,ij}$ is real at $\omega=0$. Thus in this limit the Brownian correlations are reduced to the susceptibilities of the particles to the stationary forces applied to them.

B. Role of boundaries and membranes

We now can explicitly write the expressions for the susceptibilities. We start form the plane solid wall case. From (5.4) and (3.8) we find

$$\beta_{ab,\alpha\beta} = -\frac{3(h-z_a)(h-z_b)z_az_b}{2\pi\eta h^3}\left(\frac{\delta_{\alpha\beta}}{\rho^2}-\frac{2\rho_\beta\rho_\alpha}{\rho^4}\right). \quad (5.7)$$

The components $\beta_{ab,\alpha z}$ and $\beta_{ab,zz}$ are of higher order in h/ρ and can be neglected in our approximation. Note that the susceptibility is symmetric with respect to indices a and b as well as α and β, as it should be.

We use expression (5.7) to explicitly calculate the displacement correlation function (5.6). We choose the X axis to be directed along the line connecting the particles projections to the x,y plane. In this case one finds

$$\langle[X_{a,x}(t)-X_{a,x}(0)][X_{b,x}(t)-X_{b,x}(0)]\rangle = Tt\frac{3(h-z_a)(h-z_b)z_az_b}{2\pi\eta h^3}\frac{1}{\rho^2}, \quad (5.8)$$

$$\langle[X_{a,y}(t)-X_{a,y}(0)][X_{b,y}(t)-X_{b,y}(0)]\rangle = -Tt\frac{3(h-z_a)(h-z_b)z_az_b}{2\pi\eta h^3}\frac{1}{\rho^2}, \quad (5.9)$$

$$\langle[X_{a,x}(t)-X_{a,x}(0)][X_{b,y}(t)-X_{b,y}(0)]\rangle = \langle[X_{a,y}(t)-X_{a,y}(0)][X_{b,x}(t)-X_{b,x}(0)]\rangle = 0. \quad (5.10)$$

We see that the longitudinal motion of particles is correlated in the same direction, but the transverse one (normal to the line connecting them in the x,y plane) in the opposite. This fact is in agreement with the reported experimental results [2]. We see that these two correlations are equal in value.

Let us turn to the membrane wall case. We first consider the situation when both particles are within the inner region and at a large distance from each other. We find

$$\beta_{ab,\alpha\beta} = \frac{1}{4\pi\eta}\frac{\rho_\beta\rho_\alpha}{\rho^3}, \quad (5.11)$$

$$\langle[X_{a,x}(t)-X_{a,x}(0)][X_{b,x}(t)-X_{b,x}(0)]\rangle = \frac{Tt}{2\pi\eta}\frac{1}{\rho}, \quad (5.12)$$

$$\langle[X_{a,y}(t)-X_{a,y}(0)][X_{b,y}(t)-X_{b,y}(0)]\rangle = 0, \quad (5.13)$$

$$\langle[X_{a,x}(t)-X_{a,x}(0)][X_{b,y}(t)-X_{b,y}(0)]\rangle = \langle[X_{a,y}(t)-X_{a,y}(0)][X_{b,x}(t)-X_{b,x}(0)]\rangle = 0. \quad (5.14)$$

We see that in this case the particle motion is correlated only in the longitudinal direction. These expressions are also true for the case of two particles near one membrane, provided the distance between them is much larger than their distance from the membrane.

If the particles are at different sides of the membrane we first assume that $\rho\gg h$ and $\rho\gg\gamma$. Then we can expand expression (3.13) in powers of γ/ρ to obtain

$$\beta_{ab,\alpha\beta} = \frac{1}{4\pi\eta}\left[\frac{\rho_\beta\rho_\alpha}{\rho^3}+\frac{\gamma}{\rho}\left(\frac{\delta_{\alpha\beta}}{\rho}-\frac{2\rho_\alpha\rho_\beta}{\rho^3}\right)\right], \quad (5.15)$$

which gives

$$\langle[X_{a,x}(t)-X_{a,x}(0)][X_{b,x}(t)-X_{b,x}(0)]\rangle = \frac{Tt}{2\pi\eta}\left(\frac{1}{\rho}+\frac{\gamma}{\rho^2}\right), \quad (5.16)$$

$$\langle [X_{a,y}(t)-X_{a,y}(0)][X_{b,y}(t)-X_{b,y}(0)]\rangle = \frac{Tt}{2\pi\eta}\frac{\gamma}{\rho^2}, \tag{5.17}$$

$$\langle [X_{a,x}(t)-X_{a,x}(0)][X_{b,y}(t)-X_{b,y}(0)]\rangle = \langle [X_{a,y}(t)-X_{a,y}(0)][X_{b,x}(t)-X_{b,x}(0)]\rangle = 0. \tag{5.18}$$

We note that both longitudinal and transverse correlations are present but the latter is weaker.

The above expressions do not contain any membrane characteristics (Helfrich modulus, viscosity) and hence cannot be used to find them. However, they can be used to establish whether or not the membrane is liquid and isotropic.

If the particles are at a small (in the x,y plane) distance from each other on different sides of the membrane one should use expressions (3.15) and (3.16),

$$\beta_{ab,\alpha\beta} = \frac{\delta_{\alpha\beta}}{4\pi\zeta}\ln\frac{\zeta}{\eta\gamma}, \quad \gamma \gg \rho \tag{5.19}$$

$$\beta_{ab,\alpha\beta} = \frac{\delta_{\alpha\beta}}{4\pi\zeta}\ln\frac{\zeta}{\eta\rho}, \quad \gamma \ll \rho. \tag{5.20}$$

The corresponding correlations are

$$\langle [X_{a,x}(t)-X_{a,x}(0)][X_{b,x}(t)-X_{b,x}(0)]\rangle \tag{5.21}$$

$$=\langle [X_{a,y}(t)-X_{a,y}(0)][X_{b,y}(t)-X_{b,y}(0)]\rangle = \frac{Tt}{2\pi\zeta}\log\frac{\zeta(\rho)}{\eta\gamma}, \tag{5.22}$$

$$\langle [X_{a,x}(t)-X_{a,x}(0)][X_{b,y}(t)-X_{b,y}(0)]\rangle = \langle [X_{a,y}(t)-X_{a,y}(0)][X_{b,x}(t)-X_{b,x}(0)]\rangle = 0. \tag{5.23}$$

We note that these relations contain the membrane viscosity explicitly, hence allowing one to measure it experimentally.

Let us now consider the spherical vesicle case. According to what was said at the end of Sec. IV B for large interparticle distances only the translational part of the velocity, and hence the susceptibility, is essential, so one cannot use correlations to obtain even qualitative information about the membrane. The susceptibility corresponding to this is

$$\beta_{ab,\alpha\beta} = \frac{1}{8\pi\eta_2}\frac{\delta_{\alpha\beta}+n_\alpha n_\beta}{r}. \tag{5.24}$$

We neglected the second term in (4.21) since $r \gg R$. We recall that one particle is supposed to be inside and the other outside the vesicle. The notations for the vesicle are different from those for plane geometry: r denotes the distance from the "outer" particle to the center of the vesicle, not the "inner" particle, and $\boldsymbol{n}$ is a unit vector in this direction.

At distances $r \sim a$ both parts of the flow are significant and one can use correlations to find the membrane internal viscosity, since it enters all harmonics (except the first one) of the rotational part.

VI. CONCLUSION

We derived expressions for the flows induced by stationary point forces in a fluid in the presence of membranes. The results obtained can have various applications. Particle displacement correlations can be calculated or the reaction of the fluid to the motion of a small particle immersed in it can be found.

We investigated the simplest cases of flat membranes and a nearly spherical vesicle. A system of nearly parallel membranes is realized in so-called lamellar phases of lipid solutions. Our consideration might be applicable to dilute lamellar solutions. Correlating and anticorrelating effects in the particle motion were found, which were in agreement with experimental reports for such systems [2]. As we saw in Sec. III, the presence of liquid membranes instead of solid walls affects the fluid flow in a qualitatively different way. It was shown that membranes do not suppress the flow significantly compared to solid walls. The induced velocity, and hence the particle interaction law between two parallel membranes, decays with the distance as $1/r$ which is the same as in an unconfined liquid. The decay law between two solid walls obeys $1/r^2$. It was found that there is no correlation in the motion perpendicular to the connecting line for particles between two parallel membranes. In the direction of this line, the correlation is positive and is the same as for particles in unconfined liquids.

Expressions for fluid flow both inside and outside the nearly spherical vesicle were obtained. It was found that both translational and rotational motion of the vesicle are induced. The former corresponds to the motion of the vesicle as the whole. The velocity of this motion is determined in order to equalize the force acting inside the vesicle. It appeared that the flow induced by this motion is the same as if the solid sphere were moving. The rotational part is given as a sum of spherical harmonics, and all harmonics except for the first one are significantly dependent on the membrane internal viscosity. The first harmonic represents the pure rotation of the vesicle. It does not depend on the membrane internal viscosity since the membrane is rotating as a whole. The higher harmonics contain the membrane viscosity and vanish when it approaches infinity, which corresponds to the solid membrane state. There is, consequently, a qualitative difference in the fluid flows (and hence the particle interaction law) around vesicles with solid and liquid membranes. In the former case (apart from translation) only the first spherical harmonic is induced, while in the latter case an infinite number of harmonics are present.

To summarize, we can say that the results obtained in this paper show both qualitative and quantitative differences between the induced velocity behavior laws in the presence of solid boundaries and membrane interfaces. It was found that in the presence of solid boundaries the flow is significantly suppressed. The effect of a membrane on the flow is more complicated: it damps the normal flow and the influence of the normal force component, which is similar to the solid boundary effect, but does not significantly influence the tangential flow, since the membrane is considered to be liquid. Membrane internal viscosity effects on the flow were found, which can be significant despite the small thickness of the membrane [22].

All the above suggest that results obtained in this paper can be potentially used in experimental investigations of membrane systems. The state of the membrane (liquid or solid) and the membrane internal viscosity can be tested experimentally. By measuring the particle displacement correlations or by observing the flow induced by the driven particle, one can experimentally establish the state of the membrane and measure its viscosity.

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APPENDIX A: FLAT GEOMETRY

The solution of Eqs. (3.2)–(3.4) can be written as

$$v_x = \frac{ik_x z}{2k\eta} f_1(\boldsymbol{k}) e^{kz} - \frac{ik_x z}{2k\eta} f_2(\boldsymbol{k}) e^{-kz} + f_3(\boldsymbol{k}) e^{kz} + f_4(\boldsymbol{k}) e^{-kz}, \tag{A1}$$

$$v_y = \frac{ik_y z}{2k\eta} f_1(\boldsymbol{k}) e^{kz} - \frac{ik_y z}{2k\eta} f_2(\boldsymbol{k}) e^{-kz} + f_5(\boldsymbol{k}) e^{kz} + f_6(\boldsymbol{k}) e^{-kz}, \tag{A2}$$

$$v_z = \frac{z}{2\eta} f_1(\boldsymbol{k}) e^{kz} + \frac{z}{2\eta} f_2(\boldsymbol{k}) e^{-kz} + f_7(\boldsymbol{k}) e^{kz} + f_8(\boldsymbol{k}) e^{-kz}, \tag{A3}$$

$$p = f_1(\boldsymbol{k}) e^{kz} + f_2(\boldsymbol{k}) e^{-kz}, \tag{A4}$$

with the additional conditions

$$k_x f_3 + k_y f_5 = \frac{i}{2\eta} f_1 + ikf_7, \tag{A5}$$

$$k_x f_4 + k_y f_6 = \frac{i}{2\eta} f_2 - ikf_8, \tag{A6}$$

originating from the incompressibility equation. All the f coefficients depend on the wave vector and are to be found from the boundary conditions. This solution is valid for both solid and membrane wall cases. However, the values of the coefficients will be different. Using (3.5) and (3.6) one finds for the solid walls the following cumbersome expressions:

$$f_1 = \frac{1}{2(k^2h^2 - \sinh^2 kh)} \left[F_z \left(\frac{(1-e^{-2kh})\sinh wk}{2} + kh \sinh wk - \frac{kw(1-e^{-2kh})e^{wk}}{2} - k^2 hwe^{-wk} \right) + + iF_\alpha k_\alpha \left(\frac{(1-e^{-2kh})\sinh wk}{2k} - h \sinh wk + \frac{(1-e^{-2kh})we^{wk}}{2} - khwe^{-wk} \right) \right], \tag{A7}$$

$$f_2 = \frac{F_z e^{wk}}{2} - \frac{iF_\alpha k_\alpha e^{wk}}{2k} + \frac{1}{2(k^2h^2 - \sinh^2 kh)} \left[F_z \left(\frac{(e^{2kh}-1)\sinh wk}{2} + kh \sinh wk - k^2 hwe^{wk} + - \partial_i + \frac{kw(1-e^{2kh})e^{-wk}}{2} \right) + iF_\alpha k_\alpha \left(\frac{(1-e^{2kh})\sinh wk}{2k} + h \sinh wk + \frac{(1-e^{2kh})we^{-wk}}{2} + khwe^{wk} \right) \right], \tag{A8}$$

$$f_7 = \frac{1}{4\eta} \left\{ F_z we^{-wk} - \frac{F_z \sinh wk}{k} + \frac{iF_\alpha k_\alpha we^{-wk}}{k} + \frac{iF_\alpha k_\alpha \sinh wk}{k^2} + \frac{1}{(k^2h^2 - \sinh^2 kh)} \left[F_z \left(\frac{w \sinh wk}{2} - h \sinh wk - \frac{\sinh 2kh \sinh wk}{2k} + \frac{w \sinh(2kh - wk)}{2} + khw \cosh wk \right) + iF_\alpha k_\alpha \left(\frac{\sinh wk \sinh^2 kh}{k^2} - hw \sinh wk - \frac{w \cosh wk}{2k} + \frac{w \cosh(2kh - wk)}{2k} \right) \right] \right\}, \tag{A9}$$

$$f_8 = \frac{1}{4\eta} \left\{ \frac{F_z \cosh wk}{k} - \frac{iF_\alpha k_\alpha \sinh wk}{k^2} - \frac{1}{(k^2h^2 - \sinh^2 kh)} \left[F_z \left(h \sinh wk + \frac{\sinh 2kh \sinh wk}{2k} - \frac{w \sinh wk}{2} - \frac{w \sinh(2kh - wk)}{2} - khw \cosh wk \right) + iF_\alpha k_\alpha \left(\frac{w \cosh wk}{2k} - \frac{\sinh wk \sinh^2 kh}{k^2} + hw \sinh wk - \frac{w \cosh(2kh - wk)}{2k} \right) \right] \right\}, \tag{A10}$$

$$f_3 = \frac{F_x e^{-kh}\sinh wk}{2\eta k \sinh kh} + \frac{k_x}{2\eta}\left\{\frac{iF_z w e^{-wk}}{2k} - \frac{F_\alpha k_\alpha w e^{-wk}}{2k^2} - \frac{F_\alpha k_\alpha e^{-kh}\sinh wk}{2k^3 \sinh kh} + \frac{1}{(k^2h^2 - \sinh^2 kh)}\left[F_z\left(\frac{ihw\sinh wk}{2} - \frac{ih^2\sinh wk}{2} - \frac{iw\sinh(wk-kh)\sinh kh}{2k}\right) + F_\alpha k_\alpha\left(\frac{h\sinh wk}{2k^2} - \frac{h^2\cosh kh\sinh wk}{2k\sinh kh} + \frac{hw\cosh wk}{2k} - \frac{w\cosh(kh-wk)\sinh kh}{2k^2}\right)\right]\right\}, \tag{A11}$$

$$f_4 = \frac{F_x \sinh(kh-wk)}{2\eta k\sinh kh} + \frac{k_x}{2\eta}\left\{\frac{F_\alpha k_\alpha \sinh(wk-kh)}{2k^3\sinh kh} + \frac{1}{(k^2h^2 - \sinh^2 kh)}\left[F_z\left(\frac{ih^2\sinh wk}{2} - \frac{ihw\sinh wk}{2} + \frac{iw\sinh(wk-kh)\sinh kh}{2k}\right) - F_\alpha k_\alpha\left(\frac{h\sinh wk}{2k^2} - \frac{h^2\cosh kh\sinh wk}{2k\sinh kh} + \frac{hw\cosh wk}{2k} - \frac{w\cosh(kh-wk)\sinh kh}{2k^2}\right)\right]\right\}. \tag{A12}$$

Expressions for f_5 and f_6 can be obtained from those for f_3 and f_4, respectively, by substitution of the x index with y.

In the case of membrane walls one also has to find the flow behind the membranes. Expressions (A1)–(A3) can be used here as well; however, since the liquid behind the membrane is considered to be unbounded, one has to exclude the growing exponents from all expressions; for example, for the $z<0$ region one finds

$$u_{1z} = \frac{z}{2\eta} g_1 e^{kz} + g_7 e^{kz}, \tag{A13}$$

$$u_{1x} = \frac{ik_x z}{2k\eta} g_1 e^{kz} + g_3 e^{kz}, \tag{A14}$$

$$u_{1y} = \frac{ik_y z}{2k\eta} g_1 e^{kz} + g_5 e^{kz}, \tag{A15}$$

$$p_1 = f_1 e^{kz}. \tag{A16}$$

Using the boundary conditions we discover that f_1, f_2, f_7, and f_8, and hence v_z and p, for the liquid confined between the membranes are the same as for the flow between solid walls.

The expressions for the rest of the coefficients are

$$f_3 = \frac{k_x}{2\eta}\left\{\frac{iF_z w e^{-wk}}{2k} - \frac{F_\alpha k_\alpha w e^{-wk}}{2k^2} - \frac{F_\alpha k_\alpha \sinh wk}{2k^3} + \frac{1}{(k^2h^2 - \sinh^2 kh)}\left[F_z\left(\frac{ihw\sinh wk}{2} - \frac{ih^2\sinh wk}{2} - \frac{iw\sinh(wk-kh)\sinh kh}{2k}\right) + F_\alpha k_\alpha\left(\frac{h\sinh wk}{2k^2} + \frac{hw\cosh wk}{2k} - \frac{w\sinh wk}{4k^2} - \frac{w\sinh(2kh-wk)}{4k^2} - \frac{\sinh 2kh\sinh wk}{4k^3}\right)\right]\right\}, \tag{A17}$$

$$f_4 = \frac{k_x}{2\eta}\left\{\frac{F_\alpha k_\alpha \cosh wk}{2k^3} + \frac{1}{(k^2h^2 - \sinh^2 kh)}\left[F_z\left(\frac{ih^2\sinh wk}{2} - \frac{ihw\sinh wk}{2} + \frac{iw\sinh(wk-kh)\sinh kh}{2k}\right) - F_\alpha k_\alpha\left(\frac{w\sinh wk}{4k^2} + \frac{w\sinh 2kh - wk}{4k^2} + \frac{\sinh 2kh\sinh wk}{4k^3} - \frac{h\sinh wk}{2k^2} - \frac{hw\cosh wk}{2k}\right)\right]\right\}. \tag{A18}$$

f_5 and f_6 can be obtained as was mentioned above.

APPENDIX B: SPHERICAL GEOMETRY

Explicit expressions for the boundary conditions (4.10) are

$$X_l = \frac{F\cos\theta_0}{8\pi\eta R}\left[\frac{l(l+1)}{2l-1}\left(\frac{a}{R}\right)^{l-1} - \frac{l(l+1)}{2l+3}\left(\frac{a}{R}\right)^{l+1}\right]P_l + \frac{F\sin\theta_0}{8\pi\eta R}\cos\varphi\left[\frac{(l+1)}{2l-1}\left(\frac{a}{R}\right)^{l-1} - \frac{(l+3)}{2l+3}\left(\frac{a}{R}\right)^{l+1}\right]P_l^1, \tag{B1}$$

$$Y_l = \frac{F\cos\theta_0}{8\pi\eta R}\left[\frac{l(l+1)(l+2)}{2l+31}\left(\frac{a}{R}\right)^{l+1} - \frac{l^2(l+1)}{2l-1}\left(\frac{a}{R}\right)^{l-1}\right]P_l + \frac{F\sin\theta_0}{8\pi\eta R}\cos\varphi\left[\frac{(l+2)(l+3)}{2l+3}\left(\frac{a}{R}\right)^{l+1} - \frac{l(l+1)}{2l-1}\left(\frac{a}{R}\right)^{l-1}\right]P_l^1, \quad \text{(B2)}$$

$$Z_l = \frac{F\sin\theta_0}{4\pi\eta R}\sin\varphi\left(\frac{a}{R}\right)^l P_l^1, \quad \text{(B3)}$$

where P_l and P_l^1 are Legendre polynomials defined in Sec. IV B.

The boundary conditions (4.15) and (4.16) lead to the following set of equations for the spherical vesicle:

$$\frac{F\sin\theta_0}{4\pi\eta R}\sin\varphi\left(\frac{a}{R}\right)^l P_l^1 + l(l+1)\left(\frac{R}{r}\right)^l \chi_l^{\text{in}} = l(l+1)\left(\frac{r}{R}\right)^{l+1}\chi_l^{\text{out}}, \quad \text{(B4)}$$

$$\frac{F\sin\theta_0(l+2)}{4\pi\eta_1 R}\sin\varphi\left(\frac{a}{R}\right)^l P_l^1 - \eta_1 l(l+1)(l-1)\left(\frac{R}{r}\right)^l\chi_l^{\text{in}} - \eta_2 l(l+1)(l+2)\left(\frac{r}{R}\right)^{l+1}\chi_l^{\text{out}} = \zeta\frac{l(l-1)(l+1)(l+2)}{R}\left(\frac{r}{R}\right)^{l+1}\chi_l^{\text{out}}, \quad \text{(B5)}$$

where a is the distance between the center of the vesicle and the application point.

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